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On Some Classes of Artinian Rings

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A module M is called a CS-module if every submodule of M is essential in a direct summand of M. A ring R is called CS-semisimple if every right R-module is CS. For a ring R, we show that:

(1) R is right artinian with Jacobson radical cube zero if every countably generated right R-module is a direct sum of a projective module and a CS-module.

(2) The following conditions are equivalent: (i) Every countably generated right R-module is a direct sum of a projective module and a quasicontinuous module; and (ii) every right R-module is a direct sum of a projective module and a quasi-injective module.

We describe the structure of rings in (2) and show that such a ring is not necessarily CS-semisimple.

1. INTRODUCTION

Throughout this paper, our rings are associative with identity and modules are unitary over them. A right R-module M is called a CS- (or an extending) module if every submodule of M is essential in a direct summand of M. A ring R is right CS if R_R is a CS-module. We refer to [6, 12] for details on CS-modules.

Let $\wp$ be a property of modules over a ring R (such as the property of being injective, being CS, or being a direct sum of a projective module

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and a singular module, etc.). We call R a *right $\wp$-semisimple ring* provided every right R-module satisfies $\wp$. *Left* $\wp$-semisimple rings are defined similarly, and a ring is called $\wp$-semisimple if it is right and left $\wp$-semisimple. In particular, a ring R is *right CS-semisimple* if every right R-module is CS. It is well known that right CS-semisimplicity and left CS-semisimplicity are equivalent. Hence we call a ring CS-semisimple if it is right or left CS-semisimple.

A ring R is CS-semisimple if and only if R is right and left artinian, right and left serial with $J(R)^2 = 0$ if and only if R is a direct sum of minimal right (left) ideals, and indecomposable injective right (left) ideals of composition length 2 if and only if every right (left) R-module is a direct sum of an injective module and a semisimple module (cf. [6, 13.5]). Moreover, from [11, Theorem 7] (with $M = R$), a ring R is CS-semisimple if and only if every *countably* generated right R-module is CS. In this paper, we show that this is equivalent to the condition:

($\sharp$) Every countably generated right R-module is a direct sum of a projective module and a semisimple module.

We further investigate rings satisfying a condition of type ($\sharp$) in which "semisimple" is replaced by "CS" or "quasi-continuous." Precisely, we consider the following two conditions for a right R-module M:

($\wp$) M is a direct sum of a projective module and a CS-module.

($\wp^*$) M is a direct sum of a projective module and a quasi-continuous module.

Our study of these conditions is also motivated by the work of Oshiro [14, 15] who has shown that a ring R is left artinian and QF-3 if every right R-module is a direct sum of a projective module and a singular module.

In Section 2, we show that if every countably generated right R-module satisfies ($\wp$), then R is right artinian with Jacobson radical cube zero (Theorem 5). Example 6 provides a $\wp$-semisimple ring which has Jacobson radical square nonzero, hence it is not $\wp^*$-semisimple, and consequently not CS-semisimple. It is still unknown whether a ring of Theorem 5 is right $\wp$-semisimple. On the other hand, we notice that if every finitely generated right module over a ring R is the direct sum of a projective module and an injective module, then the ring R is not necessarily right artinian (see Remark 4).

In Section 3, we characterize right $\wp^*$-semisimple rings (Theorem 7). It is shown that the class of right $\wp^*$-semisimple rings is strictly larger than the class of CS-semisimple rings (Proposition 15).

Recall that, for modules M and N, M is called N-injective if any homomorphism of a submodule of N to M can be extended to a homomorphism of N to M. A module M is said to be quasi-continuous if M is CS and for any direct summands A and B of M with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M; moreover, M is called a continuous module if M is CS and any submodule isomorphic to a direct summand of M is itself a direct summand of M. Quasi-injective modules are continuous and continuous modules are quasi-continuous. We refer to [13] for the basic properties of these modules.

For a module M, $Soc(M)$, $E(M)$, and $J(M)$ denote the socle, the injective hull, and the Jacobson radical of M, respectively. If M has finite composition length, then its length is denoted by $l(M)$.

2. CONDITION ($\wp$)

The structure of CS-semisimple rings was obtained by Dung and Smith [5] and Vanaja and Purav [19] independently.

In the first lemma we list some of the characterizations of CS-semisimple rings presented in [5, 13.5].

LEMMA 1. *For a ring R, the following conditions are equivalent*:

(a) *R is right CS-semisimple.*

(b) *Every* (*cyclic*) *right R-module is a direct sum of an injective module and a semisimple module.*

(c) *R_R is a direct sum of minimal right ideals and indecomposable injective right ideals of composition length* 2.

(d) *R is right and left artinian, right and left serial with $J(R)^2 = 0$.*

(e) *The left-handed versions of* (a), (b), *and* (c).

In this case, every right (*or left*) *R-module is a direct sum of simple modules and indecomposable injective, projective modules of length* 2.

The proof of the following lemma, due to F. L. Sandomierski, can be found in [2, Prop. 8.24].

LEMMA 2. *Let M be a nonsingular projective right R-module. If M contains a finitely generated essential submodule, then M is finitely generated.*

Next we prove the following lemma which is a crucial step in establishing the main result of this section.

LEMMA 3. *Let R be a prime right noetherian ring. If every countably generated right R-module is a direct sum of a projective module and a CS-module, then R is semisimple artinian.*

Proof. Suppose, on the contrary, that R is not artinian. As R is right noetherian and prime, R is right nonsingular. We have $Soc(R_R) = 0$, since otherwise $Soc(R_R) = R$, forcing R to be artinian, a contradiction to our assumption. We decompose the injective hull $E(R_R)$ of R into a direct sum of finitely many uniform submodules E_i:

$$E(R_R) = E_1 \oplus E_2 \oplus \cdots \oplus E_l,$$

where $E_i \cong E_j$ for any i and j. If E_1 is finitely generated then $E(R_R)$ is finitely generated. This implies that R is a simple artinian ring (see, e.g., [2, Lemma 1.19]), a contradiction to our assumption.

Therefore, E_1 is not finitely generated. Let F be a nonzero finitely generated submodule of E_1. Then $N = E_1/F$ is an infinitely generated singular right R-module. Let $\sigma[N]$ denote the full subcategory of Mod-R whose objects are submodules of N-generated modules (cf. [20]). By the hypothesis every countably generated module in $\sigma[N]$ must be CS. Hence, by [11, Theorem 7], all modules in $\sigma[N]$ are CS. Thus $N = \oplus_{\alpha \in I} N_\alpha$, where each N_α is (cyclic) uniform of length 1 or 2 (cf. [6, 13.3(h)]). It follows that the index set I is infinite. Hence we can find a countably generated proper submodule C_1 of E_1 (containing F) which is not finitely generated and such that E_1/C_1 is also infinitely generated. Pick an element $y_1 \in E_1 - C_1$ and set $C_2 = C_1 + y_1 R$. Again, pick an element $y_2 \in E_1 - C_2$ and set $C_3 = C_2 + y_2 R$. Continuing in this way, we obtain an infinite strictly ascending chain

$$C_1 \subset C_2 \subset C_3 \subset \cdots \subset C_n \subset \cdots. \tag{1}$$

We note that here each C_i is countably generated and uniform. Moreover, each C_i is not projective, since otherwise C_i containing a finitely generated essential submodule $C_i \cap R$ would be finitely generated by Lemma 2. Set $C = \oplus_{i=1}^{\infty} C_i$ and $C(m) = \oplus_{i=1}^{m} C_i$ for each $m \in \mathbb{N}$. Furthermore, set $L_i = R \cap C_i$ and $L(m) = \oplus_{i=1}^{m} L_i$. Then $L(m)$ is noetherian and essential in $C(m)$ for each $m \in \mathbb{N}$. We aim to show that the countably generated module C is CS.

We first prove that, for each $n \in \mathbb{N}$, $C(n)$ is CS. Let $n > 1$. Since $C(n)$ is countably generated we can write $C(n) = P \oplus K$ where P is a projective module and K is a CS-module. By the previous remark, $C(n)$ cannot be projective. This implies that $K \neq 0$. If $P \neq 0$ then K is not essentially in $C(n)$. Hence there exists a C_i, such that $C_i \cap K = 0$. This means that C_i can be embedded in P. But since P contains a finitely generated essential submodule $P \cap L(n)$, P is finitely generated by Lemma 2, and hence noetherian. Therefore, C_i is noetherian, a contradiction. Hence $P = 0$ and so $C(n)$ is CS.

Now we consider $C = \bigoplus_{i=1}^{\infty} C_i$. By ($\wp$), $C = P \oplus H$, where P is projective and H is CS. If $P = 0$, then we are done. Assume that P is nonzero. Then there exists a $k \in \mathbb{N}$ such that $U = P \cap C(k) \neq 0$. Since $C(k)$ is CS, U is essential in a direct summand U^* of $C(k)$. Note that U^* is also a closure of U in C. But P is also closed in the nonsingular module C and therefore $U^* \subseteq P$. By using the modular law we obtain that $P = U^* \oplus P'$. This implies that U^* is projective. As U^* contains a finitely generated essential submodule $U^* \cap L(k)$, U^* is noetherian by Lemma 2. The uniform dimension of $C(k)$ is k. Set $C(k) = U^* \oplus V$ for some submodule V. Since $U^* \neq 0$, the uniform dimension of V is at most $k - 1$. Thus there exists a C_i for some $i \in \{1, 2, \ldots, k\}$ such that $V \cap C_i = 0$. This implies that we can embed the infinitely generated module C_i in the noetherian module U^*, a contradiction. Therefore $P = 0$ and hence $C = \bigoplus_{i=1}^{\infty} C_i$ is CS.

Let i_k be the inclusion map of C_k into C_{k+1}. Then, for each $k = 1, 2, \ldots,$ i_k is a monomorphism which is not an isomorphism. Moreover, for each $n = 1, 2, \ldots$ and $0 \neq x \in C_1$, we have $i_n i_{i-1} \cdots i_1(x) = x \neq 0$. This is a contradiction to [3, Theorem 1]. Hence the injective hull of R_R must be finitely generated. Therefore, R is right artinian (cf. [2, Lemma 1.19]). ∎

Remark 4. There exist prime right noetherian non-artinian rings R for which every finitely generated right R-module is a direct sum of a projective module and an injective module.

Proof. Let R be a right and left PCI domain as constructed in [4, pp. 93–94]; i.e., R is a right and left noetherian, right and left hereditary simple V-domain which is *not a division ring*. Moreover, R/U is semisimple for any nonzero right ideal U of R. Then, by [5, 12.18], for every $m \in \mathbb{N}$, the direct sum R^m of m copies of R_R is CS. Let M be an arbitrary finitely generated right R-module. Then there exist an $n \in \mathbb{N}$ and an R-epimorphism ϕ: $R^n \to M$. Hence $M \cong R^n/\mathrm{Ker}(\phi)$. On the other hand, since R^n is CS as a right R-module, $R^n = U \oplus V$, where $\mathrm{Ker}(\phi)$ is essential in U and hence $U/\mathrm{Ker}(\phi)$ is semisimple and hence injective. Thus, M is a direct sum of a projective module V and an injective module $U/\mathrm{Ker}(\phi)$, while R is neither right nor left artinian. ∎

We now prove the main result of this section.

THEOREM 5. *For a ring R, consider the following properties*:

(a) *Every countably generated right R-module satisfies* ($\wp$).

(b) *R is right artinian and every finitely generated right R module satisfies* ($\wp$).

(c) *R is right artinian with $J(R)^3 = 0$. Moreover, for any decomposition $R_R = R_1 \oplus \cdots \oplus R_t$ with indecomposable right ideals R_i, each R_i has* *composition length at most* 3, *and the uniform dimension of each* R_i *is at most* 2.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) Let R be a ring whose countably generated right modules satisfy ($\wp$). First we show that R is right noetherian. Let E be an arbitrary essential right ideal of R and let $M = R/E$. Again let $\sigma[M]$ denote the full subcategory of Mod-R whose objects are submodules of M-generated modules. By ($\wp$), every countably generated module in $\sigma[M]$ is CS. Hence, M is noetherian by [11, Theorem 5]. Therefore, by [6, 5.15(1)], $R/Soc(R_R)$ is right noetherian. Using ($\wp$) and the arguments presented in the proof of [9, Theorem 1], we conclude that $Soc(R_R)$ is finitely generated. Hence R is right noetherian as claimed. (The above argument shows that the claim holds also if every finitely (or 2-) generated right R-module satisfies ($\wp$).)

Now we can show that R is right artinian. Suppose, on the contrary, that R is not right artinian. As R is right noetherian, if, for each prime ideal P of R, R/P is right artinian, then R is right artinian (cf. [7, 18.34B]), a contradiction. Hence there exists a prime ideal P of R such that R/P is not right artinian. Note that countably generated right modules over every factor ring of R also satisfy ($\wp$). Hence R/P is a prime right noetherian ring which satisfies ($\wp$) for its countably generated right modules. By Lemma 3, R/P is right artinian, again a contradiction. Thus R has to be right artinian.

(b) $\Rightarrow$ (c) Let $\overline{R} = R/Soc(R_R)$. Then every right $\overline{R}$-module is singular as a right R-module. By (b), every finitely generated right $\overline{R}$-module must be CS. Hence, by [11, Corollary 6], $J(\overline{R})^2 = \overline{0}$. Thus $J(R)^3 = 0$.

Write $R_R = R_1 \oplus \cdots \oplus R_t$, where each R_i is indecomposable. Hence each R_i is a local right R-module. Assume that R_i is not simple for some i, and let S be a minimal submodule of R_i. Then $V = R_i/S$ is local and not projective. By (b), V is CS. Since V is local, V is uniform. This shows that the uniform dimension of each R_i is at most 2.

Assume that there is an R_i with $l(R_i) \geq 4$. Let S be a minimal submodule in R_i. Then, as before, $V = R_i/S$ is uniform. Moreover, V is of composition length at least 3. Let T be the simple submodule of V. Then, again, V/T is CS, uniform, and of length at least 2. Let W be a submodule of V such that W/T is the minimal submodule of V/T. Hence $(V/T)/(W/T)$ is CS uniform and of length at least 1. There is a submodule Y of V containing W such that Y/W is the minimal submodule of $(V/T)/(W/T)$. This shows that Y is a (singular) uniserial module of length 3 and has a unique composition series $0 \subset T \subset W \subset Y$. The module $Y \oplus (W/T)$ is finitely generated and singular. Hence, by (b), $Y \oplus (W/T)$ is CS. But this is impossible by a result of B. L. Osofsky, see [6, 7.4]. Thus every R_i has length at most 3. ∎

We close this section with the following remarks:

(a) The ring $\mathbb{Z}/2^3\mathbb{Z}$ provides an example of a $\wp$-semisimple ring which is not CS-semisimple due to the fact that the square of its Jacobson radical is nonzero.

(b) Let $K[x]$ be the polynomial ring over a field K. Consider the factor module $M = K[x]/x^3K[x]$. Then the trivial extension R of M by K is a local commutative artinian ring with uniform dimension 3. By Theorem 5, R does not satisfy $(\wp)$ for countably generated modules, though $J(R)^2 = 0$.

3. RIGHT $\wp^*$-SEMISIMPLE RINGS

A ring R is called a right SI-ring if every singular right R-module is injective. We refer to [8] for details about SI-rings.

Obviously, a CS-semisimple ring is right $\wp$-semisimple. From the following proposition, we see that CS-semisimple rings are also $\wp^*$-semisimple.

PROPOSITION 6. *For a ring R the following conditions are equivalent*:

(i) *R is CS-semisimple*.

(ii) *Every right R-module is a direct sum of a projective module and a semisimple module*.

(iii) *Every countably generated right R-module is a direct sum of a projective module and a semisimple module*.

In particular, any CS-semisimple ring is $\wp^*$-semisimple.

Proof. (i) ⇒ (ii) Let M be an arbitrary right R-module. By Lemma 1, $M = I \oplus Q$, where I is an injective module and Q is semisimple. Since every uniform injective right R-module has composition length at most 2, without loss of generality we may assume that I is a direct sum of indecomposable injective modules I_ν each of composition length 2. Clearly, each I_ν is cyclic, and so $xR = I_\nu$ for some $0 \neq x \in I_\nu$. Then $xR \cong R_R/ann_R(x)$, where $ann_R(x)$ is the right annihilator of x in R. Since R is right CS, $R_R = U \oplus V$ where $ann_R(x)$ is essential in U. If $V = 0$, this implies that xR is a direct sum of two simple modules, a contradiction. Hence $V \neq 0$. Since xR is uniform, $U = ann_R(x)$. Therefore, $V \cong xR$, proving that I_ν is projective. Thus I is projective; i.e., (ii) holds.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i) Since every semisimple module is quasi-injective, obviously R satisfies $(\wp^*)$ for every countably generated right R-module. By Theorem 5, R is right artinian. Write $R_R = R_1 \oplus \cdots \oplus R_n$, where each R_i

is a local right R-module. Let S_i be a minimal submodule of R_i. Then, if R_i/S_i is nonzero and nonsimple, it must be projective by (ii) and by the fact that R_i is local. But this is impossible, because S_i would then split in R_i. Hence either $R_i = S_i$ or R_i/S_i is simple. Consequently, each R_i is either simple or it has composition length 2. Let R_j have composition length 2. Assume that $E(R_j) \neq R_j$. Then there is a finitely generated submodule W of $E(R_j)$ with $R_j \subsetneq W$. By (ii), W_R is a uniform projective module. By [1, 27.11], W is isomorphic to some $R_k \in \{R_1, \ldots, R_n\}$. This shows that the composition length of W_R is 2, a contradiction. Hence $E(R_j) = R_j$; i.e., R_j is injective. By Lemma 1, R is CS-semisimple, proving (i). ∎

We notice that, by Remark 4, the equivalence (i) ⇔ (iii) in Proposition 6 does not hold if the hypothesis in part (iii) is restricted to finitely generated modules.

Now we state the main result of this section in the following theorem.

THEOREM 7. *For a ring R the following conditions are equivalent*:

(I) *Every countably generated right R-module satisfies* $(\wp^*)$.

(II) *R is right artinian and every finitely generated right R-module satisfies* $(\wp^*)$.

(III) *R is a right artinian ring with Jacobson radical square zero*; $R_R = A \oplus B \oplus C$, *where* $(B \oplus C)A = BC = CB = 0$, *and B_R and C_R are nonsingular right ideals of R. Moreover*,

(i) $A_R = A_1 \oplus \cdots \oplus A_l$, *where each A_i is uniform, $E(A_i)$ is projective, and* $l(E(A_i)) \leq 2$.

(ii) *B_R is CS, and* $B_R = B_1 \oplus \cdots \oplus B_m$, *where each B_j is a uniform module of length* 1 *or* 2; *the injective hull $E(S)$ of each minimal submodule S of B_R has length* 3. *Moreover, $E(S)/S$ is a direct sum of two simple modules*; *in particular,* $E(S) = xR + yR$ *for some* $x, y \in E(S)$. *If* $B \neq 0$, *then there exist at least two* (*uniform*) *direct summands B_j and $B_{j'}$ of B with* $l(B_j) = 1$, $l(B_{j'}) = 2$, *and* $B_j \cong Soc(B_{j'})$.

(iii) $C_R = C_1 \oplus \cdots \oplus C_q$, *where each C_k is an indecomposable module of length* 1 *or* 3; *the injective hull of each minimal submodule of C_R is not projective and of length* 2. *If* $C \neq 0$, *there exist at least two C_k, say C_1 and C_2, with* $l(C_1) = 1$, $l(C_2) = 3$, *and C_1 is embedded in* $Soc(C_2)$.

(IV) *Every right R-module is a direct sum of a projective module and a quasi-injective module. In particular, R is right $\wp^*$-semisimple.*

For the proof of Theorem 7, we consider below a right artinian ring R for which every finitely generated right R-module is a direct sum of a projective module and a quasi-continuous module; i.e., $(\wp^*)$ holds for finitely generated right R-modules.

Set $\overline{R}_R = R/Soc(R_R)$. Then $\overline{R}$ is a singular right R-module, and by $(\wp^*)$, $\overline{R}_R \oplus Soc(\overline{R}_R)$ is quasi-continuous. Hence $Soc(\overline{R}_R)$ is $\overline{R}$-injective, and therefore $Soc(\overline{R}_R)$ splits in $\overline{R}_R$. Thus $\overline{R}$ is a semisimple artinian ring. This implies $J(R) \subseteq Soc(R_R)$, and hence $J(R)^2 = 0$.

Write

$$R_R = R_1 \oplus \cdots \oplus R_n, \tag{2}$$

where each R_i is indecomposable and hence local. By Theorem 5, for each R_i, u-dim$(R_i) \leq 2$ and $l(R_i) \leq 3$.

In Lemmas 8–12 below, we further investigate *this ring R and the direct summands R_i's in the above decomposition of R_R.*

LEMMA 8. *Let $R_k \in \{R_1, \dots, R_n\}$ and* u-dim$(R_k) = 2$. *Then R_k is a nonsingular right R-module.*

Proof. As u-dim$(R_k) = 2$, we have $Soc(R_k) = S \oplus T$ where S and T are minimal submodules of R_k. Since R_k/S cannot have a nonzero projective direct summand, it must be quasi-continuous and hence uniform. Note that the length of R_k/S is 2. Let T' be the image of T in R_k/S. If T' is singular, then, by $(\wp^*)$, it is easy to see that $(R_k/S) \oplus T'$ is quasi-continuous. This implies that T' is (R_k/S)-injective, so T' splits in R_k/S, a contradiction. Hence T_R is nonsingular. By a similar argument, we obtain that S_R is also nonsingular. Thus, whenever the uniform dimension of some R_k is 2, R_k is a nonsingular right R-module. ∎

LEMMA 9. *Let $R_i \in \{R_1, \dots, R_n\}$ be uniform. Then $l(R_i) \leq 2$. If $Soc(R_i)$ is singular, then R_i is injective and $l(R_i) = 2$.*

Proof. The first statement is clear, because $J(R)^2 = 0$ and R_i is a local uniform right R-module, so $l(R_i) \leq 2$. Now we assume that $S = Soc(R_i)$ is a singular submodule. As R_i is uniform, S_R is simple, and $l(R_i) = 2$. Let E_i be the injective hull of R_i. If $E_i \neq R_i$, E_i contains a finitely generated submodule U with $R_i \subset U$ such that U/R_i is simple. Hence the composition length of U is 3. Then U_R cannot be projective, since otherwise U_R must be isomorphic to some uniform right ideal of R in $\{R_1, \dots, R_n\}$ by [1, 27.11], and so $l(U_R) \leq 2$, a contradiction. The module $U_R \oplus S_R$ has uniform dimension 2. Moreover, the socle of $U_R \oplus S_R$ is singular. Write $U_R \oplus S_R = P \oplus Q$, where P is projective and Q is quasi-continuous. Clearly, Q is nonzero and $P \neq U$. Suppose that P is nonzero. Then P is uniform. Since P cannot be simple, $l(P) = 2$ (by the same argument as before that P is isomorphic to some $R_t \in \{R_1, \dots, R_n\}$). Hence $l(Q) = 2$ because $l(U_R \oplus S_R) = 4$. If $Q \cap U = 0$, then Q is embedded in S_R, a contradiction, because S_R is simple. Hence $Q \cap U \neq 0$. Consequently, $U \cap P = 0$. Therefore, P is embedded in S_R, again a contradiction, because $l(P) = 2$. Thus $P = 0$, and therefore $U_R \oplus S_R = Q$ is quasi-continuous. This means S is U-injective, and so S splits in U, a contradiction. Thus $E_i = R_i$, proving that R_i is injective and of length 2. ▮

Now we define the right ideals A, B, and C of R as follows:

1. A is the direct sum of all such $R_i \in \{R_1, \dots, R_n\}$ which are either injective, or R_i is simple with projective $E(R_i)$. In particular, by Lemmas 8 and 9, A contains the right singular ideal of R.

2. B is the direct sum of all such uniform $R_j \in \{R_1, \dots, R_n\}$; the injective hull of each $Soc(R_j)$ has length at least 3.

3. C is the direct sum of the remaining $R_k \in \{R_1, \dots, R_n\}$, i.e., the ones which are either simple or of u-dimension 2; if R_k is simple, then $l(E(R_k)) = 2$ and $E(R_k)$ is not projective.

It follows that $R_R = A \oplus B \oplus C$. Notice that A, B, or C can be zero, but all of them cannot be zero at the same time, because our ring R is assumed to be nonzero ($0 \neq 1 \in R$).

LEMMA 10. $(B \oplus C)A = BC = CB = 0$. *In particular, A is an ideal of R.*

Proof. For convenience we write

$$A = A_1 \oplus \cdots \oplus A_l, \qquad B = B_1 \oplus \cdots \oplus B_m, \qquad C = C_1 \oplus \cdots \oplus C_q,$$

where A_i, B_j, and C_k are, respectively, the R_i, R_j, and R_k as chosen above.

As mentioned before, A contains the right singular ideal of R, and hence B_R and C_R are nonsingular.

Now let $b \in B$. If $bA \neq 0$, then there exists an A_i such that $bA_i \neq 0$. Since $bA_i \subseteq B$, bA_i is nonsingular, and hence $bA_i \cong A_i$. Therefore, B has a simple submodule, and its injective hull has length 1 or 2, a contradiction to the definition of B. Thus $bA = 0$, implying $BA = 0$. Next, let $c \in C$. If $cA \neq 0$, then there exists an A_j such that $cA_j \neq 0$. Since $cA_j \subseteq C$, cA_j is nonsingular, and hence $cA_j \cong A_j$. Therefore, C has a simple module, and its injective hull is projective, a contradiction to the definition of C. Hence $cA = 0$, proving $CA = 0$.

Let $y \in B$. If $yC \neq 0$, then there is a C_k with $yC_k \neq 0$. Hence yC_k is a nonsingular submodule of B. Consequently, yC_k contains a minimal submodule Y isomorphic to one of C_k. Therefore, the injective hull of Y has length 2 (see Lemma 12 for the structure of C_R). This is a contradiction to the definition of B. Thus $yC = 0$, implying $BC = 0$. In a similar way, we get also $CB = 0$. ▮

LEMMA 11. *Let S be any minimal submodule of B. Then $l(E(S)) = 3$, and $E(S)/S$ is a direct sum of two simple modules. In particular, $E(S) = xR + yR$ for some $x, y \in E(S)$. If $B \neq 0$, then there exist at least two (uniform) direct summands B_j and $B_{j'}$ of B with $l(B_j) = 1$, $l(B_{j'}) = 2$, and $B_j \cong Soc(B_{j'})$. Moreover, B_R is CS.*

Proof. By the definition of B, we have $B = B_1 \oplus \cdots \oplus B_m$ where each B_k is given in the proof of Lemma 10. Assume that there is a B_j with $l(E(B_j)) > 3$. Then $E(B_j)$ contains submodules U, V with $U \subset V$ and $l(U) = 3$, $l(V) = 4$. The module U_R (resp. V) is not projective, since otherwise U (resp. V) is isomorphic to a member of $\{R_1, \dots, R_n\}$ which should be uniform and of length 3 (resp. 4), a contradiction to Lemma 9. By $(\wp^*)$, $U \oplus V = P \oplus Q$ where P is projective and Q is quasi-continuous. If $P \neq 0$, then P is uniform, since $U \oplus V$ is not projective and of uniform dimension 2. By Lemma 9, $l(P) \leq 2$. Hence Q is uniform and $l(Q) = 5$ or 6. Now, if $Q \cap U = 0$, Q is embedded in V, which is impossible because $l(V) = 4$. Hence $Q \cap U \neq 0$; consequently, $U \subset Q$ since Q is a closed submodule of the nonsingular module $U \oplus V$. By modularity, $Q = U \oplus (Q \cap V)$. Since $l(U) = 3$, $Q \cap V$ must be nonzero. This is a contradiction to the uniformity of Q. Therefore $P = 0$. Hence $U \oplus V$ is quasi-continuous, implying that U is $V -$ injective. Thus U splits in V, a contradiction to the uniformity of $E(B_j)$. This shows that $l(E(B_j)) = 3$, proving the first statement of this lemma.

The factor R-module $\overline{E} = E(B_j)/Soc(B_j)$ is singular and has length 2. Let $\overline{E}'$ be a simple submodule of $\overline{E}$. Then by using $(\wp^*)$ we see that $\overline{E} \oplus \overline{E}'$ is quasi-continuous. Hence $\overline{E}'$ is $\overline{E}$-injective, and so $\overline{E}'$ splits in $\overline{E}$. This shows that $E(B_j)/Soc(B_j)$ is a direct sum of two simple modules. In particular, $E(B_j) = xR + yR$ for some $x, y \in E(B_j)$. Moreover, as shown in the last part of the proof, both xR and yR are projective.

Assume that all B_j are simple. Then, for an $a \in A$, if $aB \neq 0$, there is a B_j with $aB_j \neq 0$. This shows that A contains a simple submodule $aB_j \cong B_j$. But $l(E(B_j)) = 3$, a contradiction to the definition of A. Thus $AB = 0$, and so B is a ring direct summand of R (cf. Lemma 10). In particular, B_R is injective. Thus $B = 0$, because A contains all injective R_i. Further, if there is a B_j of length 2, then the projective right R-module $Soc(B_j)$ (having injective hull of length 3) must be isomorphic to some direct summand $B_{j'}$ of B, proving the second statement of Lemma 11.

Finally, we show that B_R is CS. Let E_i be the injective hull of B_i. By the definition of B, $(E_i)_R$ is not projective. Every proper nonzero submodule X of E_i is of length 1, or 2, and hence it is cyclic, say $X = xR$ for some $x \in X$. If xR is not projective, then, by hypothesis, we can easily show that $E_i \oplus xR$ is quasi-continuous. Hence xR is E_i-injective, and so xR splits in E_i, a contradiction. Thus every proper submodule of E_i is projective.

We can write E_i in the form $E_i = uR + vR$ where uR and vR have length 2 and $uR \cap vR = Soc(E_i)$. Suppose that E_i is cyclic. Then it has to be generated by some $w \in uR + vR$. We have $w = ur + vs$ for some $r, s \in R$. If, for example, $ur \in Soc(E_i)$, then $urR \subseteq vsR \subseteq vR$, and so $wR \subseteq vR$, a contradiction. Hence we must have $urR = uR$, $vsR = vR$. Moreover, $ann_R(w) = ann_R(ur) \cap ann_R(vs)$. Therefore, $R/ann_R(w)$ contains the direct sum $[ann_R(ur)/(ann_R(ur) \cap ann_R(vs))] \oplus [ann_R(vs)/(ann_R(ur) \cap ann_R(vs))]$. Since $E_i \cong R/ann_R(w)$, one of these two direct summands must be zero, so we must have, for example, $ann_R(ur) \subseteq ann_R(w)$. But this means that $l(R/ann_R(w)) \leq l(R/ann_R(ur)) = 2$, a contradiction. Thus E_i cannot be cyclic for any $i = 1, 2, \ldots, m$. Now let V be a closed submodule of B_R. Then B/V is a cyclic nonsingular right R-module. By hypothesis, $B/V = P \oplus W_1 \oplus \cdots \oplus W_m$ where P is projective and each W_i is uniform and quasi-continuous. Since $BA = BC = 0$ (by Lemma 10), the socle of each W_i is isomorphic to that of some B_j ($j \in \{1, \ldots, m\}$). By the previous observation, each W_i cannot be injective. It follows that W_i is isomorphic to a proper submodule of $E(B_j)$ and hence projective. This shows that B/V is projective, and so V splits in B, proving that B_R is CS. ∎

LEMMA 12. *The injective hull of every simple submodule of C_R is not projective and has length 2. If $C \neq 0$, then there are C_k, say C_1 and C_2, with $l(C_1) = 1$, $l(C_2) = 3$, and C_1 is embeddable in $Soc(C_2)$.*

Proof. By the definition of C, we have $C = C_1 \oplus \cdots \oplus C_q$, where each C_k is given in the proof of Lemma 10. If $C_h \in \{C_1, \ldots, C_q\}$ and C_h is simple, then $l(E(C_h)) = 2$ and $E(C_h)$ is not projective by the definition of C.

Assume that there is a C_k with u-dim$(C_k) = 2$. Then $Soc(C_k) = S \oplus T$ where S and T are minimal submodules of C_k. Let $V = C_k/S$. Then $l(V) = 2$ and V_R is uniform and not projective. Moreover, every submodule of $E(V)$ containing V is also not projective. If $E(V) \neq V$, then there is a submodule W with $V \subset W \subseteq E(V)$ such that $l(W) = 3$. Hence, by ($\wp$*), $V \oplus W$ is quasi-continuous. This means that V splits in W, a contradiction. Thus $V = E(V)$, as desired. Similarly, $C_k/T = E(C_k/T)$. In fact, we have shown that the injective hull of every simple submodule of C_R has length 2 and is not projective.

Assume that all C_k are simple. Then, for an $a \in A$, if $aC \neq 0$, there is a C_j with $aC_j \neq 0$. This shows that A contains a simple submodule $aC_j \cong C_j$. But $E(C_j)$ is not projective, a contradiction to the definition of A. Thus $AC = 0$, and so C is a ring direct summand of R (cf. Lemma 10). In particular, C_R is injective. Thus $C = 0$, because A contains all injective R_i. Further, if there is a C_k of uniform dimension 2 (or equivalently of length

3), then any simple submodule of C_k is projective and its injective hull is not projective, of length 2. Hence it must be isomorphic to some direct summand C_h of C, proving the last statement of Lemma 12. ∎

Let R be a ring as in (III) of Theorem 7. Then $\overline{R} = R/A = \overline{B} \oplus \overline{C}$ is a ring direct sum, where $\overline{B} = (B + A)/A$ and $\overline{C} = (C + A)/A$. By [8, Chap. 3], $\overline{R}$ is a right (and left) SI-ring; i.e., every singular right (left) $\overline{R}$-module is semisimple and injective. Moreover, $\overline{R}$ is right hereditary.

LEMMA 13. *Let T be a ring such that either $T = \overline{B}$ or $T = \overline{C}$. Then every finitely generated right T-module is a direct sum of a projective module and an injective module.*

Proof. Let M be a finitely generated right T-module. Then $M = N \oplus M'$ where M' is a maximal injective submodule of M. Since T is a right SI-ring, the singular submodule of M is contained in M'. Therefore, N is nonsingular and it does not contain nonzero injective submodules. If $N = 0$, the statement is clearly true. Hence we assume $N \neq 0$, and consider below two cases (a) and (b):

(a) $T = \overline{B}$. Then T is right CS, and $T_T = T_1 \oplus \cdots \oplus T_l$ where each T_i is uniform and has the same structure as that of B_i in Lemma 10, i.e., $l(T_i) \leq 2$, and for each minimal right ideal S of T, $E(S)/S$ is the direct sum of two simple modules. Since N_T is finitely generated, there is an epimorphism φ: $T^m \to N$. Let $K = \ker \varphi$. Then $T^m/K \cong N$.

We write $T^m = V_1 \oplus \cdots \oplus V_h \oplus W$ where each V_i is isomorphic to some $T_j \in \{T_1, \ldots, T_l\}$ and $l(V_i) = 2$, and W is semisimple. Then $T^m/K = [(V_1 \oplus \cdots \oplus V_h) + K]/K + (W + K)/K$. As W is semisimple and projective, it is clear that $(W + K)/K$ is projective. Moreover, we can easily show that $K^m/K = [(V_1 \oplus \cdots \oplus V_h) + K]/K \oplus W'$ for some submodule W' of $(W + K)/K$. Hence, to get the projectivity for N, it is sufficient to show that $[(V_1 \oplus \cdots \oplus V_h) + K]/K$ is projective.

Let $H = (V_1 \oplus \cdots \oplus V_h) \cap K$. Then $[(V_1 \oplus \cdots \oplus V_h) + K]/K \cong (V_1 \oplus \cdots \oplus V_h)/H$. Since T is right hereditary, H is projective. Hence $H = H_1 \oplus \cdots \oplus H_q \oplus U$ where each H_i is of length 2 and isomorphic to some $T_j \in \{T_1, \ldots, T_l\}$, and U is semisimple. Note that $q \leq h$. After renumbering, if necessary, we can find a direct sum $V_1 \oplus \cdots \oplus V_p$ such that this is maximal with respect to the condition that $(V_1 \oplus \cdots \oplus V_p) \cap (H_1 \oplus \cdots \oplus H_q) = 0$. Then it is easy to check that

$$V_1 \oplus \cdots \oplus V_h = \left(V_1 \oplus \cdots \oplus V_p\right) \oplus \left(H_1 \oplus \cdots \oplus H_q\right). \tag{3}$$

By modularity,

$$H = H_1 \oplus \cdots \oplus H_q \oplus U', \tag{4}$$

where U' is a semisimple submodule of $(V_1 \oplus \cdots \oplus V_p)$. Since T^m/K is nonsingular, K is closed in T^m. As U' is closed in H and hence in K, U' is closed in $V_1 \oplus \cdots \oplus V_p$. Therefore, every minimal submodule of U' is also closed in $V_1 \oplus \cdots \oplus V_p$. We first prove the following:

CLAIM 1. For any minimal closed submodule L of $V_1 \oplus \cdots \oplus V_p$, $(V_1 \oplus \cdots \oplus V_p)/L$ contains a nonzero injective submodule.

We prove Claim 1 by induction on p. For $p = 1$, there is no closed minimal submodule in V_1, so the claim is true. Assume that Claim 1 holds for a $p \geq 1$. Let L be a closed minimal submodule of $V_1 \oplus \cdots \oplus V_p \oplus V_{p+1}$. If $L \subseteq V_1 \oplus \cdots \oplus V_p$, then we are done with the proof. Hence we consider the case $L \cap (V_1 \oplus \cdots \oplus V_p) = 0$ (obviously, $L \cap V_{p+1} = 0$).

If $(V_1 \oplus \cdots \oplus V_{p+1})/(L \oplus V_{p+1})$ is not nonsingular, it contains a singular minimal submodule F'. Hence the inverse image F of F' in $V_1 \oplus \cdots \oplus V_{p+1}$ is projective and u-dim$(F) = 2$. Moreover, F/V_{p+1} is embedded in $V_1 \oplus \cdots \oplus V_p$, so V_{p+1} splits in F; i.e., $F = L^* \oplus V_{p+1}$, where L^* is projective and of length 2. Since L is closed in $L^* \oplus V_{p+1}$, $(L^* \oplus V_{p+1})/L$ is nonsingular, uniform of length 3, and hence injective, proving the claim in this case.

If $(V_1 \oplus \cdots \oplus V_{p+1})/(L \oplus V_{p+1})$ is nonsingular, then it is isomorphic to the factor module of $V_1 \oplus \cdots \oplus V_p$ by a minimal closed submodule. Hence, by the induction hypothesis, $(V_1 \oplus \cdots \oplus V_{p+1})/(L \oplus V_{p+1})$ contains an indecomposable injective submodule I' (note $l(I') = 3$). Let I be the inverse image of I' in $V_1 \oplus \cdots \oplus V_{p+1}$. Then u-dim$(I) = 3$ and $l(I) = 6$. As T is right hereditary, I is projective, and hence $I = I_1 \oplus I_2 \oplus I_3$, where each I_1, I_2 are uniform of length 2, and $I_3 = V_{p+1}$. Since L is closed in I, $\bar{I} = I/L$ is nonsingular and of length 5. If L is contained in the direct sum of any two I_k, then clearly $\bar{I}$ contains a nonzero injective submodule, and we are done in this case. We consider the case that L is not contained in any of the direct sums of two I_k. Since u-dim$(\bar{I}) = 2$, $\bar{I}_1 \oplus \bar{I}_2$ is essential in $\bar{I}$. If $\bar{I}/\bar{I}_1$ contains a minimal singular submodule, then we conclude as before that I/L contains a nonzero injective submodule. Hence we are done in this case. Assume now that $\bar{I}/\bar{I}_1$ is nonsingular; consequently, $\bar{I}_1$ is closed in $\bar{I}$. This implies that $\bar{I}/\bar{I}_1$ is uniform and of length 3. Therefore, $\bar{I}/\bar{I}_1$ is injective. Now, $E(\bar{I}) = E(\bar{I}_1) \oplus E(\bar{I}_2)$. It follows that $E(\bar{I})/\bar{I}_1 = [E(\bar{I}_1)/\bar{I}_1] \oplus [(E(\bar{I}_2) + \bar{I}_1)/\bar{I}_1]$. Note that $E(\bar{I}_1)/\bar{I}_1$ is a nonzero singular module. If $\bar{I}/\bar{I}_1$ is not contained in $(E(\bar{I}_2) + \bar{I}_1)/\bar{I}_1$, then $((E(\bar{I}_2) + \bar{I}_1)/\bar{I}_1) + (\bar{I}/\bar{I}_1) = E(\bar{I})/\bar{I}_1$, and so $E(\bar{I})/\bar{I}_1$ is nonsingular, a contradiction. Hence $\bar{I}/\bar{I}_1$ is contained in $(E(\bar{I}_2) + \bar{I}_1)/\bar{I}_1$. Therefore, $\bar{I}/\bar{I}_1 = (E(\bar{I}_2) + \bar{I}_1)/\bar{I}_1$, which yields $E(\bar{I}_2) \subset \bar{I}$, proving that I/L contains a nonzero injective submodule. Thus the proof of Claim 1 is complete.

Now we return to consideration of the submodule U' in (4) above. Suppose that $U' \neq 0$. Then U' contains a minimal submodule, say S, which is closed in $V_1 \oplus \cdots \oplus V_p$. Hence $\overline{V} = (V_1 \oplus \cdots \oplus V_p)/S$ is nonsingular. By Claim 1, $\overline{V}$ contains a nonzero injective submodule, say $\overline{H}$. We can assume that $\overline{H}$ is indecomposable. Then $l(\overline{H}) = 3$. Write $U' = S \oplus Q$. Since $(V_1 \oplus \cdots \oplus V_p)/U'$ is nonsingular, $\overline{H} \cap \overline{Q} = \overline{0}$. Hence $\overline{H}$ is embedded in $(V_1 \oplus \cdots \oplus V_p)/U'$ $(\cong \overline{V}/\overline{Q})$. This is a contradiction to the fact that $(V_1 \oplus \cdots \oplus V_p)/U'$, being isomorphic to a submodule of N $(\cong T^m/K)$, does not contain a nonzero injective submodule. Thus $U' = 0$. Hence, by (3) and (4), $(V_1 \oplus \cdots \oplus V_h)/H$ is projective as desired.

(b) $T = \overline{C}$. By the properties of $\overline{C}$, we can decompose T in the form $T_T = (T_1 \oplus \cdots \oplus T_l) \oplus Z$, where each T_i is a local right T-module with u-dim$(T_i) = 2$, $l(T_i) = 3$, and Z_T is semisimple. T_T has the same structure as that of C_R in Theorem 7(iii). To verify the desired property of finitely generated nonsingular right T-modules in this case, we have to prove a statement which is similar to Claim 1.

Claim 2. Let Y be a right T-module with $Y = Y_1 \oplus \cdots \oplus Y_p$ where each Y_j is isomorphic to some $T_i \in \{T_1, \ldots, T_l\}$. Then, for each minimal submodule $L \subset Y$, the factor module Y/L contains a nonzero injective submodule.

Note that it is easy to check that every minimal submodule of Y is closed in Y. We prove Claim 2 by induction on p. For $p = 1$, Y/L is nonsingular, uniform, and of length 2. Therefore, it is injective by the property of C in (III) of Theorem 7; i.e., Claim 2 is true for $p = 1$. We assume that Claim 2 holds for a $p \geq 1$. Let L be a minimal submodule of $Y = Y_1 \oplus \cdots \oplus Y_p \oplus Y_{p+1}$ where each Y_j is isomorphic to some $T_i \in \{T_1, \ldots, T_l\}$. If $L \subset Y_{p+1}$, then Y_{p+1}/L is injective, so Y/L contains a nonzero injective submodule as desired. Hence we consider the case that L is not contained in Y_{p+1}; i.e., $L \cap Y_{p+1} = 0$. Since $Y/(L \oplus Y_{p+1}) \cong (Y_1 \oplus \cdots \oplus Y_p)/L'$ for some minimal submodule $L' \subset Y_1 \oplus \cdots \oplus Y_p$, by the induction hypothesis $Y/(L \oplus Y_{p+1})$ contains a nonzero injective submodule I'. We may assume that I' is indecomposable, and so $l(I') = 2$. As I' is nonsingular, the inverse image I of I' in Y must have uniform dimension 4 (note that u-dim$(L \oplus Y_{p+1}) = 3$). Since Y is a projective right module over the right hereditary ring T, I is projective. Moreover, I/Y_{p+1} is embeddable in $Y_1 \oplus \cdots \oplus Y_p$, and hence Y_{p+1} splits in I. Write $I = M_1 \oplus M_2$ with $Y_{p+1} = M_2$. Since $M_1 \cong I/Y_{p+1}$, M_1 is of length 3 and uniform dimension 2. Hence M_1 is isomorphic to some $T_j \in \{T_1, \ldots, T_l\}$. Now it is enough to show that I/L contains a nonzero injective submodule.

We have $M_2 \cap L = 0$. Since u-dim$(I) = 4$, there is a minimal submodule $Q \subset I$ such that $M_2 \cap (L \oplus Q) = 0$. Note that $l(I) = 6$. Hence, for

$\bar{I} = I/(L \oplus Q)$, $l(\bar{I}) = 4$ and $l(\overline{M}_2) = 3$. These show that u-dim$(\bar{I})$ can be either 2 or 3 only.

Case 1. u-dim$(\bar{I}) = 2$. In this case, $\overline{M}_2$ is essential in $\bar{I}$. As $\overline{M}_2$ is nonsingular, it follows that $\bar{I}$ is nonsingular and $\bar{I} = \overline{D}_1 \oplus \overline{D}_2$ where both $\overline{D}_i$ are of length 2 and injective.

Let $\bar{S}_1$ be the socle of $\overline{D}_1$. Set $I' = \bar{I}/\bar{S}_1$. Then $M_2' = \overline{M}_2/\bar{S}_1$ is injective and nonsingular. Hence $I' = M_2' \oplus (\overline{D}_1/\bar{S}_1)$, and I' is also injective, but it is not nonsingular. On the other hand, we can also write that $I' = [(\overline{D}_2 + \bar{S}_1)/\bar{S}_1] \oplus (\overline{D}_1/\bar{S}_1)$. Let S_2' be the socle of $D_2' = (\overline{D}_2 + \bar{S}_1)/\bar{S}_1$ $(\cong \overline{D}_2)$. If $M_2' \neq D_2'$, then $M_2' \cap D_2' = S_2'$, because otherwise $I' = M_2' \oplus D_2'$,and so $l(I')$ would be 4 which is impossible. This yields $I' = M_2' + D_2'$, implying that I' is nonsingular, a contradiction. Thus $M_2' = D_2'$ or, equivalently, $\overline{M}_2/\bar{S}_1 = (\overline{D}_2 + \bar{S}_1)/\bar{S}_1$. Hence $\overline{D}_2 \subset \overline{M}_2$, and so $\overline{M}_2 = \overline{D}_2 \oplus \bar{S}_1$, a contradiction to the fact that $\overline{M}_2$ $(\cong M_2)$ is local.

Case 2. u-dim$(\bar{I}) = 3$. It follows that $\bar{I}$ is *not* nonsingular. There is a minimal submodule $\bar{V} \subset \bar{I}$ such that $\bar{V} \oplus \overline{M}_2$ is essential in $\bar{I}$. Since $l(\bar{I}) = 4$, we must have $\bar{I} = \bar{V} \oplus \overline{M}_2$. Hence $\bar{V}$ is singular. Let V be a submodule of I that contains $L \oplus Q$ such that $V/(L \oplus Q) = \bar{V}$. Then V is nonsingular and projective with $l(V) = 3$, and u-dim$(V) = 2$. There are two possibilities: Either $V = X_1 \oplus X_2$ for uniform X_1 and simple X_2, or V is local. In the first case, $l(X_1) = 2$, and hence X_1 is injective, a contradiction because I does not contain nonzero injective submodule. Therefore, V is local, and so V is isomorphic to some $T_j \in \{T_1, \dots, T_l\}$. Hence V/L is nonsingular, uniform, and of length 2. Thus V/L is an injective submodule of I/L, as desired.

Now we can use Claim 2 to prove the statement for case (b) (i.e., the case $T = \overline{C}$) by the same way as that of (a). ∎

Proof of Theorem 7. The implication (I) ⇒ (II) is clear by Theorem 5, and (IV) ⇒ (I) is trivial.

(II) ⇒ (III) Let R be a ring satisfying (III); i.e., R is right artinian and every finitely generated right R-module is a direct sum of a projective module and a quasi-continuous module. From the discussion before Lemma 8, we have $J(R)^2 = 0$. Now, decompose R_R as in (2). Then define A, B, and C as done before Lemma 10. The first statement of (III) holds by Lemma 10, and (i) follows from the construction of A. Furthermore, (ii) and (iii) follow, respectively, from Lemmas 11 and 12. This proves (III).

(III) ⇒ (IV) Set $A = eR$, $B = fR$, and $C = gR$ where e, f, g are orthogonal idempotents with $e + f + g = 1$, the identity of R. Note that $BA = CA = BC = CB = 0$.

Let N be a nonzero right R-module. We are going to prove that N is a direct sum of a projective module and a quasi-injective module; i.e., (IV) holds.

As R is right artinian, N contains a maximal injective submodule I. It is known that $I = \oplus_{\alpha \in \Omega} I_\alpha$, where each I_α is the injective hull of a simple right R-module. Let Λ be a subset of Ω with the property that $\alpha \in \Lambda$ if and only if I_α is projective. Then the submodule $P_1 = \oplus_{\lambda \in \Lambda} I_\lambda$ is projective and injective. Write $N = P_1 \oplus M$ for some submodule M of N. In particular, M does not contain nonzero projective submodules which are injective.

We have $M = Me \oplus Mf \oplus Mg$, a group direct sum of M.

Step 1. Consider Me. If $Me \neq 0$, take any $0 \neq x \in Me$ and consider xR. Since $B \oplus C \subseteq ann_R(x)$, $xR \cong A/(A \cap ann_R(x))$. As xR does not contain nonzero injective submodules which are also projective, $A \cap ann_R(x)$ is essential in A_R. This implies that xR is singular and semisimple, and hence MeR is semisimple and singular. Moreover, eRf and eRg are subsets of the Jacobson radical of R. Hence $MeRf = MeRg = 0$, implying $(Me)R = Me(Re + Rf + Rg) = MeRe \subseteq Me$. This shows that Me is a *semisimple singular submodule* of M. We set $Q_1 = Me$.

Step 2. Consider Mf. Let $m \in M$, $r \in R$ be arbitrary elements, and write $mf.r = x_1 + x_2 + x_3$, where $x_1 \in Me$, $x_2 \in Mf$, $x_3 \in Mg$. Then, as $fre = 0$, we have $0 = m(fre) = (mfr)e = x_1 e + x_2 e + x_3 e = x_1 e = x_1$. Similarly, $x_3 = 0$. Hence $mf.r = x_2 \in Mf$. This shows that Mf is *a submodule* of M.

The factor ring $R = R/A = \overline{B} \oplus \overline{C}$ (a ring direct sum) is right SI (i.e., every singular right R/A-module is semisimple and injective (see [7, Chap. 3])). It is clear that Mf is a right module over $\overline{B}$. Let Q_2 be a maximal injective submodule of Mf. Then Q_2 contains the singular submodule of Mf which is semisimple and injective, and $Mf = Q_2 \oplus M_2$, where M_2 is a nonsingular right $\overline{B}$-module which does not contain nonzero $\overline{B}$-injective submodules. We aim to show that M_2 is projective as a right R-module.

By Zorn's lemma we find a submodule U of M_2 which is maximal with respect to the condition that $U = \oplus_{\beta \in \Gamma} U_\beta$ where each U_β is uniform projective (in Mod-R/A) and of length 2. Let T be a semisimple submodule of M_2 which is maximal with respect to the condition that $U \cap T = 0$. Hence $U \oplus T$ is essential in M_2. We aim to show that $U \oplus T = M_2$. If $U \oplus T \neq M_2$, there is a cyclic submodule $xR \subset M_2$ for which $(xR + U + T)/(U \oplus T)$ is simple. Notice that $(xR + U + T)/(U \oplus T)$ is singular. By Lemma 13 (or by the fact that $\overline{B}$ is right CS), xR is projective in Mod-(R/A), and hence $xR = V_1 \oplus \cdots \oplus V_d$ where each V_i is isomorphic to some $\overline{B}_j \in \{\overline{B}_1, \dots, \overline{B}_m\}$. There is exactly one V_i, say V_1, which is not contained in $U \oplus T$. This shows that $l(V_1) = 2$, and $U \cap V_1 = Soc(V_1)$. Since M_2 is nonsingular and V_1 is uniform, there is a finite subset $F \subseteq \Gamma$ such that $(\oplus_{\beta \in F} U_\beta) \cap V_1 = U \cap V_1 = U \cap xR$, so $(\oplus_{\beta \in F} U_\beta + V_1)/(\oplus_{\beta \in F} U_\beta)$ is simple and singular. Hence $Soc(\oplus_{\beta \in F} U_\beta)$ is essential in $\oplus_{\beta \in F} U_\beta + V_1$. On the other hand, by Lemma 13, $(\oplus_{\beta \in F} U_\beta) + V_1$ is projective as a right (R/A)-module. Hence $(\oplus_{\beta \in F} U_\beta) + V_1 = U_1' \oplus \cdots \oplus U_r'$, where each U_i' is isomorphic to some $\overline{B}_j \in \{\overline{B}_1, \dots, \overline{B}_m\}$. Hence $|F| = r$. This implies that $2r = l(\oplus_{\beta \in F} U_\beta) \geq l[(\oplus_{\beta \in F} U_\beta) + V_1]$. Therefore, $V_1 \subseteq \oplus_{\beta \in F} U_\beta$. This is a contradiction. Thus $U \oplus T = M_2$, as desired. Since T and U are clearly projective as right R-modules, so is M_2 also.

Step 3. Consider Mg. Let $m \in M$, $r \in R$ be arbitrary elements, and write $mg.r = y_1 + y_2 + y_3$, where $y_1 \in Me$, $y_2 \in Mf$, $y_3 \in Mg$. Then $0 = m(gre) = (mgr)e = y_1 e + y_2 e + y_3 e = y_1 e = y_1$. Similarly, $y_2 = 0$. Hence $mg.r = y_3 \in Mg$. This shows that Mg is *a submodule* of M.

We may consider Mg as a right $\overline{C}$-module. Let Q_3 be the maximal $\overline{C}$-injective submodule of Mg. Then $Mg = Q_3 \oplus M_3$, where M_3 is a nonsingular right $\overline{C}$-module. By Zorn's lemma we find a submodule $V \subseteq M_3$ which is maximal with respect to the condition that $V = \oplus_{\alpha \in \Omega} V_\alpha$, where each V_α is local, projective of length 3, and uniform dimension 2. Let W be a semisimple submodule of M_3 which is maximal with respect to the condition $V \cap W = 0$. Then using similar arguments as in Step 2, we obtain $M_3 = V \oplus W$. Since V and W are clearly projective as right R-modules, so is M_3 also.

Summing up the preceding three steps, we obtain $N = (P_1 \oplus M_2 \oplus M_3) \oplus (Q_1 \oplus Q_2 \oplus Q_3)$, where $P_1 \oplus M_2 \oplus M_3$ is projective. Moreover, Q_1, Q_2, and Q_3 are quasi-injective and relatively injective to each other (in fact, there are no nonzero homomorphisms between them, because $Q_1 = Q_1 e$, $Q_2 = Q_2 f$, $Q_3 = Q_3 g$, and e, f, g are orthogonal idempotents in R). Hence $Q_1 \oplus Q_2 \oplus Q_3$ is quasi-injective. This completes the proof of (IV). ∎

For a ring R, we consider the following conditions, where $sl(M)$ denotes the Loewy length of a module M:

(a) Every countably generated right R-module is a direct sum of a projective module and a quasi-continuous module.

(b) Every right R-module is a direct sum of indecomposable modules M_i with $l(M_i) \leq 3$ and $sl(M_i) \leq 2$.

Then, by Theorem 7, (a) ⇒ (b). We can verify that all M_i in (b) are 2-generated. However, in general, a 2-generated indecomposable module L with $l(L) = 3$ may have Loewy length 3 also.

We notice that a ring R of Theorem 7 is a ring-direct sum of A, B, C if R is right nonsingular. The following consequence of Theorem 7 gives an answer when R is exactly right nonsingular.

COROLLARY 14. *For a ring R the following conditions are equivalent*:

(a) *Every countably generated right R-module is a direct sum of a projective module and an injective module.*

(b) *R is right artinian and every finitely generated right R-module is a direct sum of a projective module and an injective module.*

(c) *R is a right nonsingular, right artinian ring with Jacobson radical square zero*; $R = A \oplus B \oplus C$ *is a ring-direct sum where* A, B, C *are as in Theorem* 7; A *is a right nonsingular CS-semisimple ring.*

(d) *Every right R-module is a direct sum of a projective module and an injective module.*

Proof. (a) ⇒ (b) and (d) ⇒ (a) are clear. For (b) ⇒ (c), let N be a singular right ideal of R. Since N_R is finitely generated, N_R is injective by (b). Hence $N = 0$, proving that R is right nonsingular. Hence we can show that $AB = BA = CB = BC = AC = CA = 0$, where A, B, C are defined as in Theorem 7. It follows that $R = A \oplus B \oplus C$ is a ring-direct sum. Since A_A is a direct sum of injective right ideals of length less than or equal to 2 and minimal right ideals, A is CS-semisimple by Lemma 1.

(c) ⇒ (d) Let M be a right R-module. Then $M = M_1 \oplus M_2$ where M_1 is a right A-module and M_2 is a right $(B \oplus C)$-module. By the last statement of Lemma 1, M_1 is a direct sum of a projective module and an injective module. For M_2 we use similar arguments as those in the proof of (III) ⇒ (IV) of Theorem 7 to get that M_2 is a direct sum of a projective module and an injective module. Thus (d) holds. ▮

PROPOSITION 15. *There exist right CS, right* $\wp$*-semisimple rings which are not CS-semisimple.*

Proof. Consider the following ring

$$R = \begin{bmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{bmatrix},$$

where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. Then R is right and left artinian, right and left hereditary, SI, right serial, and $J(R)^2 = 0$. Moreover, it is easy to show that R is right CS. Further, R is not left serial. Hence, by Lemma 1, R is *not CS-semisimple*.

It is clear that

$$E = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{bmatrix}$$

is the injective hull of $e_{11}R$. Since $l(E_R) = 3$, we conclude that the injective hull $E(S)$ of any minimal right ideal S of R has length 3, and

$E(S)/S$ is the direct sum of two simple modules. Therefore, R is a ring of Theorem 7(III) with $A = C = 0$. ∎

One can also use results from [17, 18] to obtain the conclusion of Proposition 15 for the ring $\left[\begin{smallmatrix}\mathbb{R} & \mathbb{C}\\ 0 & \mathbb{C}\end{smallmatrix}\right]$ as pointed out by the referee.

Although the ring $R = \left[\begin{smallmatrix}\mathbb{R} & \mathbb{C}\\ 0 & \mathbb{C}\end{smallmatrix}\right]$ is right CS, the right R-module $R \oplus R$ is not CS, because if $(R \oplus R)_R$ were CS, it would follow from a result of [16] that R is CS-semisimple.

If we take R to be the ring $\left[\begin{smallmatrix}\mathbb{C} & \mathbb{C}\\ 0 & \mathbb{R}\end{smallmatrix}\right]$. Then R is left and right artinian and nonsingular. However, R is not right CS. Write $R = e_{11}R \oplus e_{22}R$ where $e_{11}R$ is a local right R-module with u-dim$(e_{11}R) = 2$, $l(e_{11}R) = 3$, and $e_{22}R$ is simple. Moreover, we can verify that the injective hull of every minimal right ideal of R has length 2. (See the discussion about this ring in [10].) Thus R is a ring of Theorem 7 with $A = B = 0$.

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